Time-Space Lower Bound for Parity Learning

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Machine Learning Project Presentation
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Learning Parity

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$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i \bmod 2$$

$$\xrightarrow[\mathbf{a} \leftarrow_R \{0,1\}^n]{\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle}$$

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$$\downarrow$$

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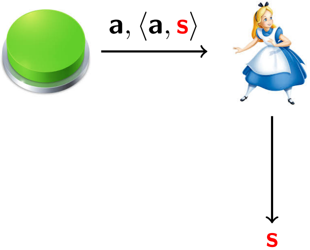
With **large memory**:

Gather n vectors $\mathbf{a}$ into an invertible $\mathbf{A}$,

$\longrightarrow$ obtain $(\mathbf{A}, \mathbf{A} \cdot \mathbf{s})$

Then use **Gaussian Elimination:**

$\longrightarrow \sim n$ samples and n^2 memory



Learning Parity: Two strategies

A secret vector $\mathbf{s} = (s_1, \ldots, s_n) \in \{0,1\}^n$.

With **large memory**:
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$\longrightarrow$ obtain $(\mathbf{A}, \mathbf{A} \cdot \mathbf{s})$
Then use **Gaussian Elimination:**

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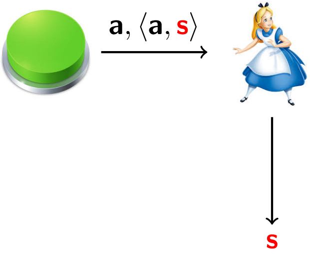
With **a lot of samples:**
Wait for a sample $\mathbf{a} = (1, 0, \ldots, 0)$;
read $s_1 = \langle \mathbf{a}, \mathbf{s} \rangle$.

$\longrightarrow \sim n \cdot 2^n$ samples and n memory



Time-Memory Tradeoff for Learning Parity

- Those are the **only two** strategies!

Main Theorem ([Raz, FOCS'16] , informal)

Any algorithm that learns parity either:

- Uses $\sim \mathbf{n^2}$ **memory**,

OR

- Uses an **exponential number of samples**.

Time-Memory Tradeoff for Learning Parity

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Main Theorem ([Raz, FOCS'16])

For all $c < 1/20$, there exists an $\alpha > 0$ such that
any program using

- $\leq \mathbf{2^{\alpha n}}$ **samples**
- $\leq \mathbf{cn^2}$ **memory**

only succeeds with probability $\leq \mathcal{O}(2^{-\alpha n})$.

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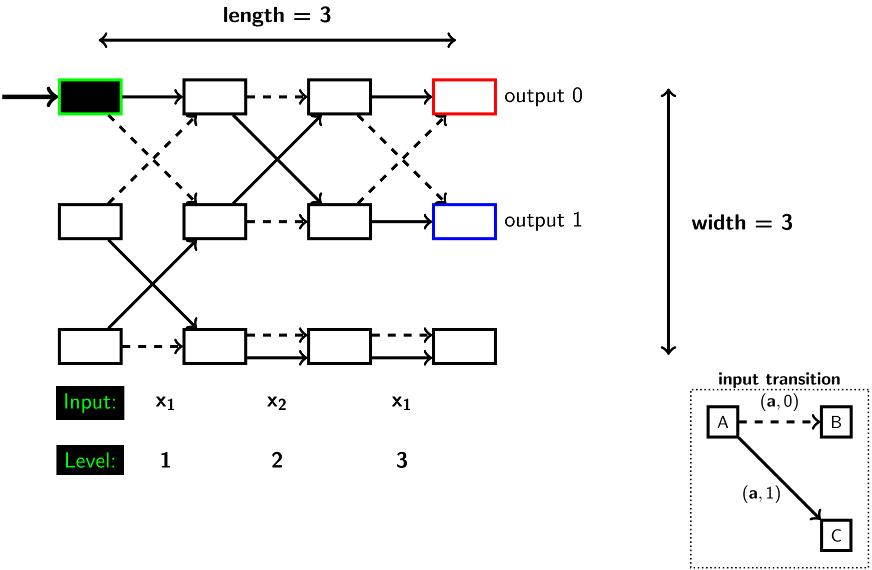
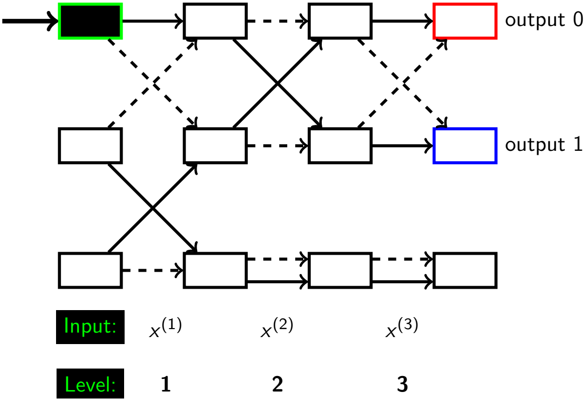
- Proof?

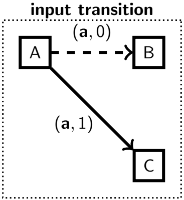
  Need a computational model for **bounded memory**.

Branching Programs



$$\text{Input: } x^{(1)} = (\mathbf{a}_1, 0)$$





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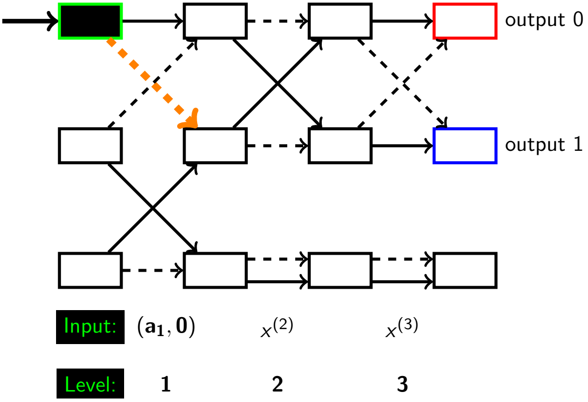
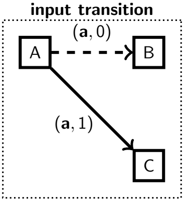
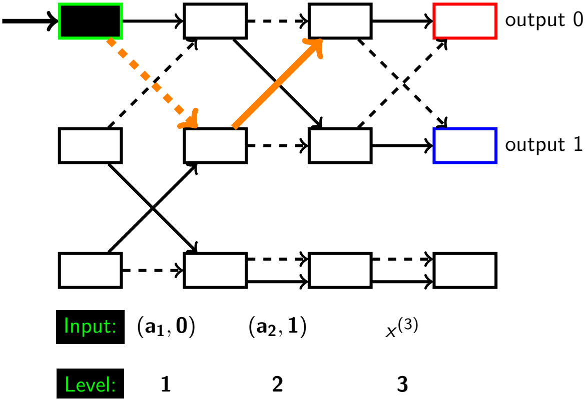
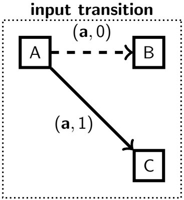




Branching Programs

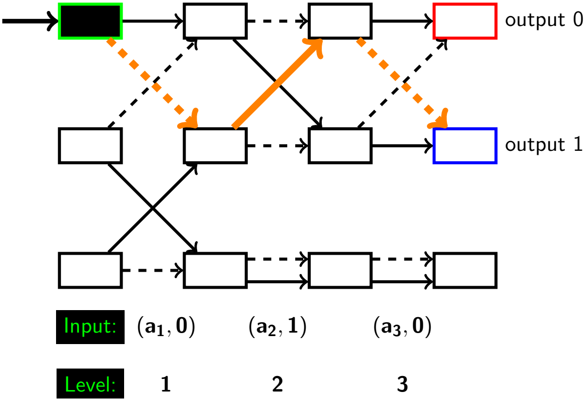
Input: $x^{(2)} = (\mathbf{a}_2, 1)$

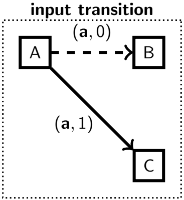




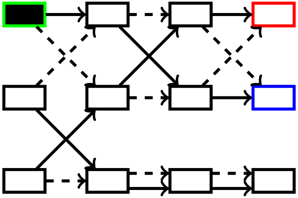
Branching Programs

$$\text{Input: } x^{(3)} = (\mathbf{a}_3, 0)$$





- Idea: embed **structure** into the computation.



Affine Branching Programs

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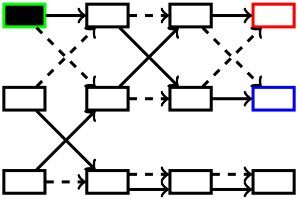
- Memorize **subspace** where **s** lives in **nodes**.



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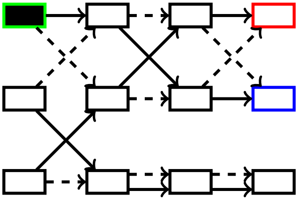
- Associate **nodes** with **affine subspaces.**
  - Starting node is **the whole space**.

$$E_{start} = \{0,1\}^n.$$



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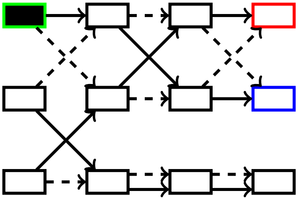
  - Edge (u, v) for input $(\mathbf{a}, b)$,

    Restricts **s** to a **smaller subspace**.

    $$E_v \supseteq E_u \cap \{\mathbf{s} \mid \langle \mathbf{a}, \mathbf{s} \rangle = b\}$$



Proof Outline

The proof goes in two steps:

Step 1, easy-ish

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Step 2, much harder

Any **Branching Program** can be simulated by an **Affine Branching Program**.

Summary

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Follow-up works: ([Kol-Raz-Tal, STOC'17], [Raz, FOCS'17], [Garg-Raz-Tal, STOC'18]...)

- Generalize lower-bound for **larger class of problems**
  - **Sparse** parities
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Thank you for your attention